Fig. 1a). This condition yields an additional relationship between the two parameters (M_1 and M_2), closes the system (1.1)-(1.3), and makes the dependence of the coefficients on M_1 single-valued for all flow deviation angles as long as they are not too large. This is why the dependence of the optimum permeability coefficient of the wall on M in supersonic flow is universal [1].

For flow that is supersonic but in the transonic regime, a unique dependence no longer exists for the optimum permeability coefficient; it varies along the length of the pipe wall in each case and differs for each experiment. To illustrate this fact, we give the example of the calculation of a gas flow around a slender wedge at zero angle of attack with a velocity slightly greater than the sonic velocity. The half-angle of the wedge is $\theta = 1°22'$.

Figure 4 shows how R_{opt}, normalized to $\sqrt{M_1^2 - 1}$, varies as a function of M_1 in the region above the wedge behind the compression shock. The calculations have been carried out according to the exact gasdynamic equations and tables [4].

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AN EXACT SOLUTION FOR THE END EFFECT OF A WING OF FINITE SIZE IN SUPERSONIC FLOW

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UDC 533.69

The problem of supersonic flow over a thin wing of finite size, examined in a linear approximation, reduces to solving the wave equation for the velocity potential. The condition that the flow does not penetrate the wing surface is then carried to the base plane, and in the remainder of this plane (outside the projection of the wing) certain conditions are imposed on the gasdynamic parameters of the flow. The solution of the problem is given in [1] when the velocity potential is determined via the normal derivative in the base plane Φ'_η, and outside the wing projection we have the condition that the potential goes to zero. The gasdynamic flow parameters (pressure, downwash outside the wing) obtained from this solution take on physically invalid infinite values in the vicinity of the subsonic leading edge. Expressions are given in [2] for the velocity potential and its derivatives in terms of the first and second derivatives of the potential in the base plane, which allows one to apply additional boundary conditions and obtain a solution of the flow problem in which the gasdynamic flow parameters are in a class of bounded functions.

This paper derives formulas for calculating the gasdynamic flow parameters in the case when the velocity potential is determined [2] via the first derivative Φ'_η and the second derivative $\Phi''_{\eta\xi}$ (the surface curvature in the incident stream direction) in the base plane, and in the part of the base plane outside the wing projection the condition of continuity of the derivative Φ'_ξ (pressure) is applied.

1. The velocity potential at the point M(x, y, z) lying in the perturbed region above the wing is found via the normal derivative Φ'_η in the base plane $\eta = 0$ from the formula [1]

$$\Phi = -\frac{1}{\pi}\iint\limits_{s+\sigma} \Phi'_\eta(\xi, \zeta)\,\varphi\, d\xi\, d\zeta, \tag{1.1}$$

where $\varphi = r^{-1}$; $r = \sqrt{(x-\xi)^2 - (z-\zeta)^2 - y^2}$; $(s+\sigma)$ is the region of dependence of the point M in the plane $\eta = 0$ (Fig. 1). Part of the region of dependence $s(COO_1D_1M_0C)$ coincides with the

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projection of the wing S on the base plane. The part $\sigma(O_1DD_1O_1)$ belongs to the perturbed region Σ of the plane $\eta = 0$, lying outside the wing projection in the zone of the influence of the lateral edge. In the region S the value Φ'_η is determined by the wing geometry: $\Phi'_\eta(\xi, \zeta) \sim \alpha(\xi, \zeta)$ [$\alpha(\xi, \zeta)$ is the slope of the wing surface to the base plane in the direction of the x axis]. In the region Σ (in the gap) the derivative Φ'_η is unknown beforehand and is to be determined.

Following integration with respect to the variable ξ and allowing for the relations

$$\int \varphi\, d\xi = \ln\{[(x-\zeta)+r]/\sqrt{(z-\zeta)^2+y^2}\},$$

$$\left(\int \varphi\, d\xi\right)_{\xi=\rho(\zeta)} = 0$$

[$\xi = \rho(\zeta) = x - \sqrt{(z-\zeta)^2 + y^2}$ is the line of intersection of the characteristic cone with the vertex at the point M with the plane $\eta = 0$] we can write Eq. (1.1) for the potential as

$$\Phi = \frac{1}{\pi}\int_L \left(\Phi'_\eta \int \varphi d\xi\right)_{\xi=\psi^-(\zeta)} d\zeta + \frac{1}{\pi}\iint_{s+\sigma} \Phi''_{\eta\xi}\int \varphi d\xi ds \tag{1.2}$$

[$\xi = \psi^-(\zeta)$ is the equation of the boundary of the perturbed region $L(COO_1D)$]. In deriving Eq. (1.2) we used the condition that the derivative Φ'_η is continuous at the lateral edge (boundary of the regions S and Σ) and the condition of convergence.

According to Eq. (1.2), taking account of the relation $\partial\varphi/\partial x = -\partial\varphi/\partial\xi$ the derivatives of the potential Φ are represented in the form

$$\Phi'_x = -\frac{1}{\pi}\int_L (\Phi'_\eta\varphi)_{\xi=\psi^-(\zeta)} d\zeta - \frac{1}{\pi}\iint_{s+\sigma}\Phi''_{\eta\xi}\varphi ds; \tag{1.3}$$

$$\Phi'_z = \frac{1}{\pi}\int_L\left(\Phi'_\eta\int\frac{\partial\varphi}{\partial z}d\xi\right)_{\xi=\psi^-(\zeta)} d\zeta + \frac{1}{\pi}\iint_{s+\sigma}\Phi''_{\eta\xi}\int\frac{\partial\varphi}{\partial z}d\xi\, ds, \tag{1.4}$$

where $\int\frac{\partial\varphi}{\partial z}d\xi = \frac{(x-\xi)(z-\zeta)}{r[(z-\zeta)^2+y^2]}$. In differentiating Eq. (1.2) the terms of differentiation with respect to the variable limits, depending on the variables x and z, go to zero.

For points M(x, y, z) in the zone with no annular effect (the region $\sigma = 0$) in Eqs. (1.3) and (1.4) the values of Φ'_η and $\Phi''_{\eta\xi}$ are given by the wing geometry and the gasdynamic flow parameters are computed directly from these formulas.

In the case of the influence of the ends on the section of the line L coincident with the projection of the wing leading edge COO_1, the derivative is defined by the wing geometry ($\Phi'_\eta \sim \alpha[\psi^-(\zeta), \zeta]$), in the section O_1D (the wake of the leading characteristic surface) $\Phi'_\eta|_{O_1D} = 0$, in part of the region of dependence s the derivative $\Phi''_{\eta\xi}$ is defined by the wing geometry, and in σ the value $\Phi''_{\eta\xi}$ is unknown beforehand and to be determined.

The problem of flow over a wing allowing for the end effect (calculating Φ'_η in the region Σ) was solved in [1] under the condition $\Phi(M\in\Sigma) = 0$ [the potential Φ is represented in the form of Eq. (1.1)]. The solution of this integral equation in a characteristic coordinate system (x_1, z_1) with the origin at the point O_1 (see Fig. 1) coinciding with the point of transition of the supersonic leading edge to subsonic (lateral) flow is written in the form

$$\Phi'_y(x_1, z_1) = -\frac{1}{\pi\sqrt{z_1 - f(x_1)}}\int_{\psi(x_1)}^{f(x_1)}\alpha(x_1,\zeta_1)\frac{\sqrt{f(x_1)-\zeta_1}}{z_1-\zeta_1}d\zeta_1 \tag{1.5}$$

where $z_1 = \psi(x_1)$, $z_1 = f(x_1)$ are equations of the leading and lateral edges, and $\Phi'_\eta(x_1, \zeta_1) = \alpha(x_1, \zeta_1)$ is a function specified by the wing geometry.

The second derivative $\Phi''_{yx}(M\in\Sigma)$ in the gap was obtained in agreement with [2] from solution of the integral equation $\Phi'_x(M\in\Sigma) = 0$ [with Φ'_x written in the form of Eq. (1.3)], this being a condition for continuity of pressure in the region Σ:

$$\Phi''_{yx}(x_1, z_1) = -\frac{1}{\pi\sqrt{z_1 - f(x_1)}}\left\{\int_{\psi(x_1)}^{f(x_1)}\alpha'_\xi(x_1,\zeta_1)\frac{\sqrt{f(x_1)-\zeta_1}}{z_1-\zeta_1}d\zeta_1 + \alpha[x_1, \psi(x_1)]\cdot\psi'_{x_1}(x_1)\frac{\sqrt{f(x_1)-\psi(x_1)}}{z_1-\psi(x_1)}\right\} \tag{1.6}$$

[($\alpha'_\xi(x_1, \zeta_1) = \Phi''_{\eta\xi}(x_1, \zeta_1)$ is a function specified by the wing geometry, and $\alpha[x_1, \psi(x_1)] = \Phi'_\eta[x_1, \psi(x_1)]$ is the angle of attack at the wing leading edge].

For the velocity potential Φ in [1] and for Φ'_x in [2], using values of Φ'_η, $\Phi''_{\eta\xi}$ in the region Σ, theorems were proved about the end effect, according to which the potential Φ and the derivative Φ'_x for points M lying above the wing are written as

$$\Phi = -\frac{1}{\pi}\iint\limits_{s_1} \Phi'_\eta \varphi ds, \quad \Phi'_x = -\frac{1}{\pi}\int\limits_{L_1} (\overline{\Phi}'_\eta \varphi)_{\xi=\psi-(\zeta)} d\zeta - \frac{1}{\pi}\iint\limits_{s_1} \Phi''_{n\xi}\varphi ds,$$

where s_1 is part of the region of dependence s of the point M on the projection of the wing in the base plane, limited by the line of intersection of the characteristic cone with the base plane CM_0D_1 (D_1 is the point of intersection of this line with the lateral edge), the characteristic D_1C_1 and the intercept of the leading edge CC_1 (the line L_1). The values of $\Phi'_\eta, \Phi''_{\eta\xi}$ in the region s_1 and on L_1 are determined by the wing geometry.

2. As can be seen from Eq. (1.5), in solving the problem of flow over a thin wing of finite span, according to [1] the derivative $\Phi'_y(M \in \Sigma)$ (the flow downwash at the gap), as the point M comes close to the lateral edge tends to infinity as $r^{-1/2}$ when $r \to 0$. From the solution we also obtain [1] the fact that the derivative $\Phi'_x(M \in \Sigma)$ (pressure) as the point M draws near to the lateral edge from the wing side has the same order of singularity (apart from the case when the lateral edge lies in the plane parallel to the incident stream velocity).

It follows from Eq. (1.6) that the second derivative $\Phi''_{yx}(M \in \Sigma)$ from the solution obtained according to [2], as the point M moves close to the lateral edge has the same singularity as does the first derivative $\Phi'_y(M \in \Sigma)$ of the solution of [1].

We now analyze the behavior of the first derivatives Φ'_y, Φ'_x, Φ'_z (of the gasdynamic parameters of the flow) of the solution [2]. Without distorting the basic properties of the solution, in order to reduce the cumbersome calculations in performing the required operations of integrating the single and double integrals an analysis was done on the example of solving the problem of flow over a flat plate for which the behavior of the gasdynamic parameters of the solution is well known [1, 3].

For planar flow in the wing projection region on the base plane $S\ \Phi'_\eta(M \in S) = \alpha = \text{const}$, $\Phi''_{\eta\xi}(M \in S) = 0$ and Eq. (1.6) is transformed to

$$\Phi''_{yx}(M \in \Sigma) = -\frac{\alpha\psi'_{x_1}(x_1)\sqrt{f(x_1) - \psi(x_1)}}{\pi\sqrt{z_1 - f(x_1)}\,[z_1 - \psi(x_1)]}. \tag{1.6'}$$

Without restricting generality in order to simplify the calculations we considered flow over a plate whose leading edge is a straight line perpendicular to the incident stream velocity, and the lateral edge is also a straight line. For this wing the equations of the leading and lateral edges will be, respectively $z_1 = -x_1$, $z_1 = k_1x_1$ ($1 \leqslant k_1 \leqslant \infty$; $k = 1$ when the lateral edge coincides with the direction of the incident stream velocity, $k_1 = \infty$ when the subsonic lateral edge becomes sonic, coinciding with the bow characteristic $O_1D(x_1 = 0)$], and Eq. (1.6') takes the form

$$\Phi''_{yx}(M \in \Sigma) = \frac{\alpha\sqrt{k_1+1}\sqrt{x_1}}{\pi\sqrt{z_1 - k_1x_1}\,(x_1 + z_1)}. \tag{1.6''}$$

The second derivative $\Phi''_{yx}(M \in \Sigma)$ goes to zero on the bow characteristic and tends to infinity in the vicinity of the lateral edge.

In Eqs. (1.2)-(1.6) for all the derivatives the subscripts denote the direction of differentiation, and they coincide with the direction of the coordinate axes of the original system (x, y, z) (which is not a characteristic). In order for the first derivative $\Phi''_{yx} \times (M \in \Sigma)$ to be determined from the value of the second derivative $\Phi'_y(M \in \Sigma)$, it is appropriate to rewrite Eq. (1.6") in (x, z) coordinates:

$$\Phi''_{yx}(M \in \Sigma) = \frac{\alpha\sqrt{x-z}}{\sqrt{2}\,\pi\sqrt{z - kx\,x}}. \tag{1.6'''}$$

The coordinate origin of the (x, z) system coincides with the origin of the characteristic system (x_1, z_1), located at the corner point of transition of the wing leading edge to the

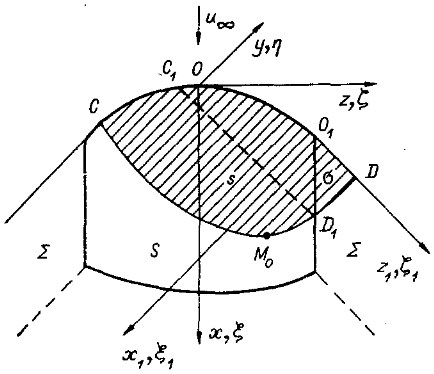

Fig. 1

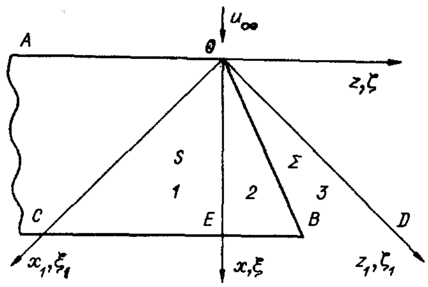

Fig. 2

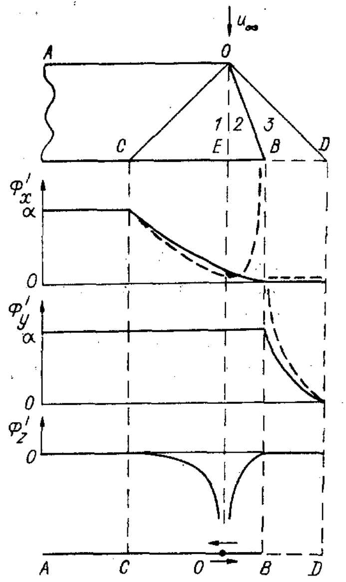

Fig. 3

lateral edge (Fig. 2). The coordinates are connected by the relations $x = (x_1 + z_1)/\sqrt{2}$, $z = (z_1 - x_1)/\sqrt{2}$, and the tangents of the angles of inclination in the lateral edge equations $z = kx$, $z_1 = k_1 x_1$ by the dependence $k = (k_1 - 1)/(k_1 + 1)$. When the direction of the lateral edge changes from coinciding with the incident flow velocity to coinciding with the bow characteristic OD the parameters k_1, k vary in the limits $1 \leqslant k_1 \leqslant \infty$, $0 \leqslant k \leqslant 1$.

After integrating both parts of Eq. (1.6''') with respect to x we find the first derivative to be

$$\Phi_y'(M \in \Sigma) = -\frac{\alpha}{\sqrt{2}\,\pi}\left\{\frac{1}{\sqrt{k}}\arcsin\frac{(1+k)z - 2kx}{(1-k)z} + \arcsin\frac{(1+k)x - 2z}{(1-k)x}\right\} + az + b. \tag{2.1}$$

The flow about the corner point of the flat plate possesses conical symmetry, and Eq. (2.1) can be written in the form

$$\Phi_y'(M \in \Sigma) = -\frac{\alpha}{\sqrt{2}\,\pi}\left\{\frac{1}{\sqrt{k}}\arcsin\frac{(1+k) - 2k\frac{x}{z}}{1-k} + \arcsin\frac{(1+k) - 2\frac{z}{x}}{1-k}\right\} + a_1\frac{z}{x} + b, \tag{2.2}$$

where $k \leqslant z/x \leqslant 1$; $0 < k < 1$. The derivative $\Phi_y'(M \in \Sigma)$ is a function bounded in the entire region Σ, including on the characteristic lines: the bow characteristic ($z = x$), and the lateral characteristic ($z = kx$).

The solution (2.2) is arbitrary in the choice of the two constants a_1 and b and these can be used to impose the additional conditions corresponding to the physical picture of the flow: on the bow characteristic

$$\Phi_y'(M \in \Sigma) = -\frac{\alpha}{2\sqrt{2}}\left(\frac{1}{\sqrt{k}} - 1\right) + a_1 + b = 0; \tag{2.3}$$

and on the lateral characteristic

$$\Phi_y'(M \in \Sigma) = \frac{\alpha}{2\sqrt{2}}\left(\frac{1}{\sqrt{k}} - 1\right) + ka_1 + b = \alpha. \tag{2.4}$$

These conditions are met for all $0 < k < 1$. For the sonic edge ($k = 1$), when it coincides with the bow characteristic, conditions (2.3) and (2.4) become contradictory.

There is a special case when the lateral edge is parallel to the incident stream velocity ($k = 0$). On integration of Eq. (1.6''') with $k = 0$ we have

$$\Phi_y''(M \in \Sigma) = \frac{\alpha\sqrt{2}}{\pi}\left\{\sqrt{\frac{x}{z} - 1} - \operatorname{arctg}\sqrt{\frac{x}{z} - 1}\right\} + a_1\frac{z}{x} + b. \tag{2.5}$$

Hence it can be seen that for such a wing the flow downwash in the vicinity of the lateral edge ($z = 0$) tends to infinity.

The behavior of the gasdynamic parameters Φ_x', Φ_z', according to Eqs. (1.3) and (1.4) on the properties of the derivatives Φ_η', $\Phi_{\eta\xi}''$, given in region S by the wing geometry and in region Σ determined by Eqs. (1.6''') and (2.2). In Eqs. (1.3) and (1.4) we substitute the corresponding values of the derivatives Φ_η', $\Phi_{\eta\xi}''$ and perform the rather laborious process of computing the integrals which can be obtained in final form. It is more convenient to perform the computation in the characteristic variables (x_1, z_1). The region of influence of the lateral edge, bounded by the bow characteristics OD ($\xi_1 = 0$), OC ($\zeta_1 = 0$) is divided into three: region 1 on the wing, adjoining the wing zone AOC, where there is no end effect, bounded by the characteristic OC and the line OE ($\zeta_1 = \xi_1$); region 2 on the wing bounded by the line OE and the lateral edge OB ($\zeta_1 = k_1\xi_1$); and region 3, coincident with region Σ, and bounded by the edge OB and the characteristic OD (see Fig. 2). The point M belonging to these regions we will designate below as M_1, M_2, and M_3.

After performing all the computations for Φ_x', Φ_z', we obtain the compact expressions:

$$\Phi_x'(M_1, M_2) = \frac{\alpha}{\pi}\left\{\frac{\pi}{2} + \arcsin\frac{(k_1x_1 - z_1) - (k_1+1)z_1}{k_1(x_1+z_1)}\right\}; \quad \Phi_x'(M_3) = 0; \tag{2.6}$$

$$\Phi_z'(M_1, M_2) = -2\alpha\sqrt{\frac{k_1+1}{k_1-1}}\ln\frac{\sqrt{(k_1-1)z_1} + \sqrt{k_1x_1 - z_1}}{\sqrt{k_1|x_1 - z_1|}}, \quad \Phi_z'(M_3) = 0. \tag{2.7}$$

According to Eq. (2.6), at the boundary with the region that does not influence that lateral edge (the line OC, $z_1 = 0$) Φ_x' coincides with the appropriate solution for a flat plate of infinite size: $\Phi_x'|_{z_1=0} = \alpha$. On the lateral edge, as must follow from the statement of the problem, $\Phi_x'|_{z_1 = k_1x_1} = 0$. On the wing, within the perturbed lateral edge region in the pressure distribution there are no singularities, including also the line $\zeta_1 = \xi_1$, where

$$\Phi_x'|_{z_1=x_1} = \frac{\alpha}{\pi}\left\{\frac{\pi}{2} - \arcsin\frac{2}{k_1(k_1+1)}\right\}.$$

The solid line on Fig. 3 is a qualitative picture of the pressure distribution at the section x = const, and the broken line is the pressure distribution from [1]. Figure 3 also shows a qualitative picture of the behavior of the derivative Φ_y' in the gap, according to Eq. (2.2), taking account of conditions (2.3) and (2.4) (solid line) and the solution of Eq. (1.5) on the basis of [1] (broken line).

The derivative Φ_z' characterizes the transverse overflow in the base plane $\eta = 0$. According to Eq. (2.7) in the gap, just as on the flat plate of infinite size, there is no transverse overflow [$\Phi_z'(M_3) = 0$]. The solution in the perturbed lateral edge region on the wing $\Phi_z'(M_1, M_2)$ is joined continuously with the solution in the gap and on the infinite flat plate (Fig. 3). On the line $\zeta_1 = \xi_1$ which passes through the point of bending of edges in the direction of the incident flow, Φ_z' has a singularity. On this line is located a singularity of the longitudinal vortex type, which rarely does not influence the components Φ_y', Φ_x' (Fig. 3). In the case where the lateral edge is parallel to the incident flow velocity, the singularity coincides with the edge, which is reflected in the derivative Φ_y' at the edge [Eq. (2.5)].

Thus, if in the case of an edge effect involving $\Phi_{\eta\xi}''$ as the governing parameter we start from the condition $\Phi_x' = 0$ in the base plane outside the wing projection, then the gasdynamic flow parameters (the first derivatives of the potential) at the subsonic edge assume finite values, in contrast with the solution of [1]. Reference [1] dealt with symmetric (unseparated) flow over a wing of finite size. In the solution examined the author imposed the condition of attachment on the subsonic edges, corresponding to the picture of separated flow over a wing of finite size.

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